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Simultaneous design of AWC and nonlinear controller for uncertain nonlinear systems under input saturation

Muntazir Hussain[1] | Muhammad Rehan[1] | Choon Ki Ahn[2] | Keum-Shik Hong[3] | Najam us Saqib[1]

[1]Department of Electrical Engineering, Pakistan Institute of Engineering and Applied Sciences, Islamabad, Pakistan

[2]School of Electrical Engineering, Korea University, Seoul, South Korea

[3]Department of Cogno-Mechatronics Engineering and School of Mechanical Engineering, Pusan National University, Busan, South Korea

Correspondence
Choon Ki Ahn, School of Electrical Engineering, Korea University, Seoul 02841, South Korea.
Email: hironaka@korea.ac.kr

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Summary

This article addresses a novel technique for the simultaneous design of a robust nonlinear controller and static anti-windup compensator (AWC) for uncertain nonlinear systems under actuator saturation and exogenous $\mathcal{L}_2$ bounded input. The system is presumed to have locally Lipschitz nonlinearities, time-varying uncertainties (appearing both in the linear as well as nonlinear dynamics and both in the state in addition to the output equations), and external norm-bounded inputs both in the state and the output equations. Several bilinear matrix inequality–based conditions are derived to simultaneously design the robust nonlinear controller and AWC gains for uncertain nonlinear systems by employing the Lyapunov functional, reformulated Lipschitz property, uncertainty bounds, linear parameter-varying approach, modified local and global sector conditions, iterative linear matrix inequality algorithm, convex optimization procedure, and $\mathcal{L}_2$ gain minimization. The proposed multiobjective AWC-based dynamic robust nonlinear controller guarantees the mitigation of saturation effects, robustness against time-varying parametric norm-bounded uncertainties, the asymptotic stability of the closed-loop nonlinear system under zero external disturbances, and the attenuation of disturbance effects under nonzero external disturbances. The effectiveness of the proposed AWC-based dynamic robust nonlinear controller synthesis scheme is illustrated by simulation examples.

KEYWORDS

anti-windup compensator, dynamic robust nonlinear controller, input saturation, linear parameter-varying technique, uncertain nonlinear systems

1 | INTRODUCTION

The synthesis of multiobjective robust controllers for uncertain nonlinear systems has become an increasingly challenging task as requirements and specifications for controller designs have become more rigorous. Standard tools and methods are available to design a robust linear controller for linear systems; however, all physical systems are subjected to operational constraints induced by technological, physical, or even security considerations. These constraints are mostly associated with the amplitude limitation of the input actuators. Therefore, a controller design without consideration of these constraints may lead to objectionable or even terrible process performance, such as loss of stability. In most cases, accurately dimensioned actuators saturate even under usual operating conditions because of, for example, perturbations, disturbances, load variations, and set-point changes. Under actuator saturation, the controller cannot provide the pre-specified design performances. The closed-loop system may progress to some undesired operating conditions with the possibility that, when the actuator saturation finishes, it may not return to the ordinary operating condition. Actuator saturation is a limitation of practical systems because actuators cannot transmit an unconstrained energy signal, producing saturation or windup effects.[1-7]

Integral windup compensation methods are broadly classified into two types. The first method is the two-step methodology, in which a dynamic controller is tuned without considering actuator saturation effects, and then, an additional anti-windup compensator (AWC) is augmented in the closed-loop control system to mitigate the windup effects. This approach occupies a noteworthy portion of the literature.[2-4] However, this technique is often criticized owing to its conservatism and poor closed-loop performance. To achieve the improved closed-loop performance and stability, a second method, called the one-step approach, is employed, which simultaneously designs the controller and AWC to attain the desired response. In this approach, the information of saturation nonlinearity is directly included in the synthesis of a multiobjective controller. The multiobjective controller is responsible for tracking, robustness, and windup effect compensation. This methodology usually leads to a better closed-loop performance against input saturation (see, for example, other works[7-10]); however, the derivation of the design conditions in this case is a highly challenging task due to the computation of several control and anti-windup gains.

Anti-windup compensation for linear systems is a well-established subject. Both the one-step[9,10] and two-step approaches[3] have been widely studied to compensate the saturation effects in stable and unstable linear and nonlinear control systems. Numerous techniques and tools are available for designing AWCs for linear control systems along with successful applications to laboratory setups, such as a disk drive,[11] ball-and-beam system,[12] and wireless networks.[13] Recently, many researchers have shown an active interest in designing AWCs for nonlinear systems, and many design methodologies are proposed in the literature. We can refer, for example, to the AWC design for Euler-Lagrange nonlinear systems,[14] feedback-linearizable nonlinear systems,[15] dynamic inversion-based nonlinear systems,[16] Lipschitz nonlinear plants,[2,4] rational nonlinear systems,[5,17] uncertain nonlinear systems,[18] Lur'e nonlinear models,[19] Takagi-Sugeno systems,[20] and nonlinear time-delay prototypes.[3] Most recently, anti-windup designs for linear parameter-varying (LPV) systems were proposed.[21-23] Morabito et al[14] proposed a method of constructing an AWC for Euler-Lagrange nonlinear systems, which ensures the global asymptotic and local exponential stabilization of saturated closed-loop systems. A dynamic AWC for feedback-linearizable nonlinear systems is presented in the work of Yoon et al[15] as an extension of the work of Park and Choi.[24] The approach of Herrmann et al[16] generalized the AWC framework[25,26] to the case of saturated feedback-linearizable nonlinear affine systems controlled by employing nonlinear dynamic inversion (NDI) control methodologies. In the work of Rehan and Hong,[2] linear matrix inequality (LMI)–based local and global dynamic AWC designs are proposed for Lipschitz nonlinear systems. Meanwhile, in the work of Hussain et al,[4] the results of the work of Rehan and Hong[2] were extended for exponential synthesis and an $\mathcal{L}_2$ exponentially stable regional AWC for nonlinear systems. The one-step approach-based simultaneous design of a dynamic controller and an AWC for nonlinear systems with Lipschitz nonlinearity and actuator saturation was investigated by Rehan et al.[8] Wang et al[17] investigated both dynamic and static AWC schemes for rational nonlinear systems using linear-fractional representation, while da Silva et al[5] suggested a static AWC design for multivariable rational nonlinear systems by means of differential algebraic representation. An LMI-based framework is proposed to synthesize the output feedback controller for Lur'e nonlinear systems[19] and Takagi-Sugeno systems[20] based on Lyapunov theory and the modified sector condition. The controller design problem for switched and sampling systems was investigated in the works of Ma et al[27] and Yang et al,[28] respectively. The technique in the work of Hussain and Rehan[3] proposes AWC synthesis for constrained nonlinear time-delay systems by employing a delay-range-dependent methodology. Furthermore, an internal model control–based AWC and decoupled and equivalent decoupled-based anti-windup compensation architectures were schematized for Lipschitz nonlinear time-delay systems.

Over the past decade, the control community has paid close attention to a specific type of nonlinear system called the Lipschitz nonlinear system (see the works of Rehan et al[2-4,8] and the references therein). However, the Lipschitz condition does not represent the unique characteristics of nonlinear systems and may cause conservatism. The controller or AWC synthesis based on the conventional Lipschitz condition can lead to infeasible results for larger Lipschitz constants. Most recently, this conservatism has been removed by introducing the reformulated Lipschitz condition.[7,29] This reformulated Lipschitz condition using the LPV approach represents all attributes of the nonlinear systems. To the best of the authors' knowledge, an LPV-based simultaneous design of a robust nonlinear controller and an AWC for nonlinear systems with time-varying parametric uncertainties, actuator saturation, and exogenous inputs has not been studied in the previous works.

In light of the aforementioned literature, this research is dedicated to the simultaneous design of a robust nonlinear controller and an AWC for uncertain locally Lipschitz nonlinear systems under actuator saturation nonlinearity and exogenous $\mathcal{L}_2$ bounded inputs. By using Lyapunov stability, Lipschitz reformulation, the local region of interest, uncertainty bounds, LPV theory, global and local sector conditions, the iterative LMI (ILMI) algorithm, the convex optimization procedure, and $\mathcal{L}_2$ gain minimization, several bilinear matrix inequality (BMI) conditions are derived to find a nonlinear controller and static AWC gains. The proposed AWC-based dynamic nonlinear controller ensures the mitigation of saturation effects, robustness against time-varying parametric norm-bounded uncertainties, and stability of the overall closed-loop system. The key contributions of this study are summarized as follows.

(1) A novel technique for the simultaneous design of the robust nonlinear controller and AWC for uncertain nonlinear systems under actuator saturation, parametric uncertainty, and exogenous $\mathcal{L}_2$ inputs is proposed for the first time to the best of our knowledge.
(2) The multiobjective robust nonlinear controller is designed by considering the less conservative LPV-based formulation of the locally Lipschitz nonlinearities.
(3) Parametric uncertainties (in the linear as well as nonlinear parts) are considered for a comprehensive design framework in contrast to the existing techniques.
(4) An approach for finding the controller and static AWC gain matrices by employing the recursive convex procedures is suggested.

The efficiency of the proposed methodology is verified via simulation examples of a nonlinear system and a one-link flexible robot.

The remaining section of this paper is organized as follows. Section 2 formulates the system and the proposed AWC-based dynamic nonlinear controller description and presents preliminary results. In Section 3, the proposed AWC synthesis is developed and BMIs are derived to obtain the nonlinear controller and AWC gain matrices. Section 4 gives the application results. Finally, concluding remarks are provided in Section 5.

Notation. Standard notations are used throughout this article. For any real matrix $X \in \Re^{n\times n}$, X^T and X^{-1} signify the transpose and inverse of X, respectively. For any symmetric matrix $S = S^T$, $S > 0$ and $S \geq 0$ represent the positive definite and positive semidefinite matrices, respectively. $He\{M\} = M + M^T$. $\Omega(\mu P) = \{x(t) \mid x^T(t)P \leq 1\}$ represents an ellipsoidal region. For any two vectors $z(t) = (z_1(t), z_2(t), \dots, z_n(t))^T$ and $\overline{z}(t) = (\overline{z}_1(t), \overline{z}_2(t), \dots, \overline{z}_n(t))^T$, an augmented vector obtained by the combination of $z(t)$ and $\overline{z}(t)$ can be represented as $z^{\overline{z}_i(t)}(t) = (\overline{z}_{,1}(t), \dots, \overline{z}_i(t), z_{i+1}(t), \dots, z_n(t))^T$. Correspondingly, we can characterize $z^{0_i}(t) = (0, \dots, 0, z_{i+1}, (t), \dots, z_n(t))^T$. The symbols $\|w(t)\|$ and $\|w(t)\|_2$ signify the Euclidian norm and $\mathcal{L}_2$ norm of $w(t)$. The identity matrix is indicated by I. diag$\{x_1(t), x_2(t), \dots, x_n(t)\}$ indicates a block diagonal matrix. The input saturation nonlinearity, for the control input $u(t) \in R^m$, is indicated by $\mathcal{N}_{sat}(u(t))$.

2 | SYSTEM DESCRIPTION

Considering the state-space realization of uncertain nonlinear systems given by

$$
\begin{aligned}
\dot{x}(t) &= \widehat{A}_p(t)x(t) + f_p(t,x) + \Delta f_p(t,x) + B_{pu}\mathcal{N}_{sat}(u_c(t)) + B_{pw}w(t), \\
y(t) &= \widehat{C}_y(t)x(t) + f_y(t,x) + \Delta f_y(t,x) + D_{yw}w(t), \\
z(t) &= \widehat{C}_z(t)x(t) + f_z(t,x) + \Delta f_z(t,x) + D_{zu}\mathcal{N}_{sat}(u_c(t)) + D_{zw}w(t),
\end{aligned}
\tag{1}
$$

where $x(t) \in \Re^n$, $u_c(t) \in \Re^m$, $\mathcal{N}_{sat}(u_c(t)) \in \Re^m$, $y(t) \in \Re^q$, $z(t) \in \Re^s$, and $w(t) \in \Re^l$ represent the plant states, unsaturated control inputs, saturated control inputs, measured and performance outputs, and external inputs (which may contain a reference, disturbance, noise, etc), respectively. The nonlinear vector functions are denoted as $f_p(t,x) = B_{pf}f(t,x) \in \Re^n$, $f_y(t,x) = D_{yf}f(t,x) \in \Re^q$, and $f_z(t,x) = D_{zf}f(t,x) \in \Re^s$, and the uncertain nonlinear vector functions are represented by $\Delta f_p(t,x) = \Delta B_{pf}(t)f(t,x) \in \Re^n$, $\Delta f_y(t,x) = \Delta D_{yf}(t)f(t,x) \in \Re^q$, and $\Delta f_z(t,x) = \Delta D_{zf}(t)f(t,x) \in \Re^t$. The system matrices are $\widehat{A}_p(t) = (A_p + \Delta A_p(t)) \in \Re^{n\times n}$, $\widehat{C}_y(t) = (C_y + \Delta C_y(t)) \in \Re^{p\times n}$, and $\widehat{C}_z(t) = (C_z + \Delta C_z(t)) \in \Re^{t\times n}$, where A_p, B_{pu},

C_y, C_z, D_{zu}, B_{pw}, D_{yw}, and D_{zw} are known constant matrices and $\Delta A_p(t)$, $\Delta C_y(t)$, $\Delta C_z(t)$, $\Delta B_{pf}(t)$, $\Delta D_{yf}(t)$, and $\Delta D_{zf}(t)$ are unknown matrices containing the time-varying uncertain parameters. The uncertainties satisfy

$$\begin{bmatrix} \Delta A_p(t) & \Delta B_{pf}(t) \\ \Delta C_y(t) & \Delta D_{yf}(t) \\ \Delta C_z(t) & \Delta D_{zf}(t) \end{bmatrix} = \begin{bmatrix} M_{pa} & M_{pf} \\ M_{yc} & M_{yf} \\ M_{zc} & M_{zf} \end{bmatrix} F(t) N, \tag{2}$$

where M_{pa}, M_{yc}, M_{zc}, M_{pf}, M_{yf}, M_{zf}, and N are the known constant matrices of suitable dimensions. The matrix $F(t)$ is unknown time varying for all $t > 0$ and fulfills

$$F^T(t) F(t) \leq I. \tag{3}$$

Consider a robust nonlinear controller, along with the static AWC, represented by

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + f_{bc}(t,x) + B_{cy} y(t) + E_1 \zeta_z(u_c(t)), \\ u_c(t) &= C_c x_c(t) + f_{dc}(t,x) + D_{cy} y(t) + E_2 \zeta_z(u_c(t)), \end{aligned} \tag{4}$$

where $x_c(t) \in \mathfrak{R}^n$, $u_c(t) \in \mathfrak{R}^m$, and $\zeta_z(u_c(t)) \in \mathfrak{R}^m$ indicate the controller states, unsaturated control inputs, and dead-zone function, respectively. The vector functions $f_{bc}(t,x) = B_{bcf} f(t,x) \in \mathfrak{R}^n$ and $f_{dc}(t,x) = D_{dcf} f(t,x) \in \mathfrak{R}^n$ indicate the nonlinear dynamics of the controller. The controller matrices A_c, B_{bcf}, B_{cy}, C_c, D_{dcf}, and D_{cy} and the static AWC matrices E_1 and E_2 need to be designed to provide the desired closed-loop performance and compensation against saturation effects.

Assumption 1. The exogenous input $w(t)$ is $\mathcal{L}_2$ bounded; that is, we have

$$\lambda^{-1} \|w(t)\|_2^2 \leq 1. \tag{5}$$

Exogenous input $w(t)$ can include the measurement noise and/or the disturbance input. The energy bound on the exogenous input is generally ubiquitous and unknown. In most control design problems, the exogenous input is taken as bounded for each time instant during the plant operation time, which is bounded. In particular, in optimization-based design methods (H_∞ control, the linear-quadratic regulator, $\mathcal{L}_2$ gain minimization, and linear-quadratic-Gaussian control, etc), we need to assume that the defined system inputs are bounded (for example, by norm or stochastic characteristics). An unbounded input or disturbance can drive the system away from the desired operation point. In the $\mathcal{L}_2$ gain minimization, the energy from input to output in the closed-loop system is minimized. When one input has unbounded energy, the norm will not make sense anymore because it can result in infinite output energy; otherwise, an unbounded control action is (theoretically) needed for dealing with an unbounded input. In either case, the boundedness assumption on the input or disturbance is a practical step for realizing a control formulation problem.

Assumption 2. For any $x(t), \bar{x}(t) \in \mathrm{B}_f$, where $\mathrm{B}_f = \{x(t) \mid |x^T(t)| \leq \kappa_h\}$ for $h = 1, 2, 3, \ldots, n$, the nonlinear function $f(t,x)$ with scalar κ_f satisfies

$$\|f(t,x) - f(t,\bar{x}))\| \leq \kappa_f \left\| \left(x(t) - \bar{x}(t) \right) \right\|. \tag{6}$$

Note that Assumption 2 considered herein assumes that the nonlinear dynamics is locally Lipschitz rather than the conservative global Lipschitz condition. The Lipschitz nonlinear function in (6) can be reformulated as follows (see the works of Wang et al[7] and Zemouche and Boutayeb[29] for details):

$$\begin{aligned} & f(t,x) - f\left(t,\bar{x}\right) = \sum_{i=1}^{r} \sum_{j=1}^{n} f_{ij} \left(e_n(i) e_n^T(j) \right) \left(x(t) - \bar{x}(t) \right), \\ & \forall x(t), \bar{x}(t) \in \mathrm{B}_f \subseteq R^n, \forall \Theta_f \in \mathcal{W}_f, \end{aligned} \tag{7}$$

where the function $f_{ij} : R^n \times R^n \to R$ (for all $i = 1, 2, \ldots, r$, and $j = 1, 2, \ldots, n$) with the bound $\underline{\lambda}_{f_{ij}} \leq f_{ij} \leq \overline{\lambda}_{f_{ij}}$ is defined as

$$f_{ij} = \begin{cases} 0, & if\, x_j = \bar{x}_j, \\ \dfrac{f_i\left(x^{\bar{x}_{j-1}}\right) - f_i\left(x^{\bar{x}_j}\right)}{x_j - \bar{x}_j}, & if\, x_j \neq \bar{x}_j, \end{cases} \tag{8}$$

where $\underline{\lambda}_{f_{ij}}$ and $\overline{\lambda}_{f_{ij}}$ represent the lower and upper bounds, respectively. For $f(0,x) = 0$, the nonlinear function $f(t,x)$ can be symbolized as

$$f(t,x) = \sum_{i=1}^{r} \sum_{j=1}^{n} f_{ij} \left(e_n(i) e_n^T(j) \right) x(t), \tag{9}$$

$$f_{ij} = \begin{cases} 0, & \text{if } x_j = 0, \\ \frac{f_i(x^{0_{j-1}}) - f_i(x^{0_j})}{x_j}, & \text{if } x_j \neq 0. \end{cases} \tag{10}$$

By defining $\Theta_f = \sum_{i=1}^{r} \sum_{j=1}^{n} f_{ij}(e_n(i)\, e_n^T(j))$, we obtain $f(t,x) = \Theta_f x(t)$. Since the function f_{ij} is bounded, the matrix Θ_f belongs to a restricted convex set $\mathcal{W}_f$ with the vertices set specified as

$$\mathcal{V}_{\mathcal{W}_f} = \left\{ \Psi^f \in \mathrm{R}^{n\times n} \mid \Psi_{ij}^f \in \left\{ \underline{\lambda}_{f_{ij}}\, \overline{\lambda}_{f_{ij}} \right\} \right\}. \tag{11}$$

The subsequent system matrices are defined as

$$\begin{aligned}
\overline{A} &= \begin{bmatrix} A_p + B_u D_{cy} C_y & B_u C_c \\ B_{cy} C_y & A_c \end{bmatrix}, \ \overline{B}_{pf} = \begin{bmatrix} B_{pf} + B_u D_{cy} D_{yf} \\ B_{cy} D_{yf} \end{bmatrix}, \\
\Delta\overline{A}(t) &= \begin{bmatrix} \Delta A_p(t) + B_u D_{cy} \Delta C_y(t) & 0 \\ B_{cy} \Delta C_y(t) & 0 \end{bmatrix}, \ \Delta\overline{B}_{pf}(t) = \begin{bmatrix} \Delta B_{pf}(t) + B_u D_{cy} \Delta D_{yf}(t) \\ B_{cy} \Delta D_{yf}(t) \end{bmatrix}, \\
\overline{B}_{\zeta_z} &= \begin{bmatrix} 0 & B_u \\ I_m & 0 \end{bmatrix}, \ \overline{B}_w = \begin{bmatrix} B_w + B_u D_{cy} D_{yw} \\ B_{cy} D_{yw} \end{bmatrix}, \ \overline{B}_u = \begin{bmatrix} -B_u^T & 0 \end{bmatrix}^T, \ \overline{I}_{n\theta} = \begin{bmatrix} \theta I_n & 0 \end{bmatrix}, \\
\overline{I}_{nc} &= \begin{bmatrix} 0 & I_{nc} \end{bmatrix}^T, \ \overline{E} = \begin{bmatrix} E_1^T & E_2^T \end{bmatrix}^T, \ \overline{C}_z = \begin{bmatrix} C_z + D_{zu} D_{cy} C_y & D_{zu} C_c \end{bmatrix}, \\
\overline{D}_{zf} &= \left(D_{zf} + D_{zu} D_{cy} D_{yf} \right), \ \Delta\overline{C}_z(t) = \begin{bmatrix} \Delta C_z(t) + D_{zu} D_{cy} \Delta C_y(t) & 0 \end{bmatrix}, \\
\Delta\overline{D}_{zf}(t) &= \left(\Delta D_{zf}(t) + D_{zu} D_{cy} \Delta D_{yf}(t) \right), \ \overline{I}_m = \begin{bmatrix} 0 & I_m \end{bmatrix}, \\
\overline{D}_w &= \begin{bmatrix} D_{zw} + D_{zu} D_{cy} D_{yw} \end{bmatrix}, \ \overline{C}_c = \begin{bmatrix} D_{cy} C_y & C_c \end{bmatrix}, \ \overline{D}_{cy} = D_{cy} D_{yf}, \\
\Delta\overline{C}_c(t) &= \begin{bmatrix} D_{cy} \Delta C_y(t) & 0 \end{bmatrix}, \ \Delta\overline{D}_{cy}(t) = D_{cy} \Delta D_{yf}.
\end{aligned} \tag{12}$$

Taking $\chi(t) = [x(t) \ \ x_c(t)]^T$ as an augmented state vector, the closed-loop nonlinear system attained by connecting the nonlinear system (1) and controller (2) is given by

$$\begin{aligned}
\dot{\chi}(t) &= \left(\tilde{A}\left(\Theta_f\right) + \Delta\tilde{A}\left(\Theta_f, t\right) \right) \chi(t) + \tilde{B}_{\zeta_z} \zeta_z(u_n(t)) + \tilde{B}_w w(t), \\
z(t) &= \left(\tilde{C}_z\left(\Theta_f\right) + \Delta\tilde{C}_z\left(\Theta_f, t\right) \right) \chi(t) + \tilde{D}_{\zeta_z} \zeta_z(u_n(t)) + \tilde{D}_w w(t), \\
u_c(t) &= \left(\tilde{C}_c\left(\Theta_f\right) + \Delta\tilde{C}_c\left(\Theta_f, t\right) \right) \chi(t) + \overline{I}_m \overline{E} \zeta_z(u_n(t)) + \widehat{D}_{ucw} w(t),
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
\tilde{A}\left(\Theta_f\right) &= \overline{A} + \overline{B}_{pf} \overline{I}_{n\theta} - \overline{B}_u D_{dcf} \overline{I}_{n\theta} + \overline{I}_{nc} B_{bcf} \overline{I}_{n\theta}, \ \tilde{B}_w = \overline{B}_w, \\
\Delta\tilde{A}\left(\Theta_f, t\right) &= \Delta\overline{A}(t) + \Delta\overline{B}_{pf}(t) \overline{I}_{n\theta}, \ \tilde{B}_{\zeta_z} = \overline{B}_{\zeta_z} \overline{E} + \overline{B}_u, \\
\tilde{C}_z\left(\Theta_f\right) &= \overline{C}_z + \overline{D}_{zf} \overline{I}_{n\theta} + D_{zu} D_{dcf} \overline{I}_{n\theta}, \ \tilde{D}_w = \overline{D}_w, \\
\Delta\tilde{C}_z\left(\Theta_f, t\right) &= \Delta\overline{C}_z(t) + \Delta\overline{D}_{zf}(t) \overline{I}_{n\theta}, \ \tilde{D}_{\zeta_z} = \left(D_{zu} \overline{I}_m \overline{E} - D_{zu} \right), \\
\tilde{C}_c\left(\Theta_f\right) &= \overline{C}_c + \overline{D}_{cy} \overline{I}_{n\theta} + D_{dcf} \overline{I}_{n\theta}, \ \widehat{D}_{ucw} = D_{cy} D_{yw}, \\
\Delta\tilde{C}_c\left(\Theta_f, t\right) &= \Delta\overline{C}_c(t) + \Delta\overline{D}_{cy}(t) \overline{I}_{n\theta}.
\end{aligned} \tag{14}$$

LPV theory is an extension of gain-scheduling techniques. In gain-scheduling methods, several linear time-invariant controllers are designed for a parameterized family of linearized models. This technique leads to reasonable results if the variations of the parameters are adequately slow. The reformulated Lipschitz property provides an alternate approach to transform the nonlinear systems into LPV systems, in which the system matrix is affine in a parametric form. The proposed LPV technique requires 2^{n^2} LMIs to be solved for the n-dimensional nonlinear state vector. It should be noted that the LPV approach provides less restrictive design conditions than the conventional methods by employing the bounds on the nonlinear functions.

Our objective is to propose a novel method for the simultaneous design of dynamic controller matrices A_c, B_{bc}, B_{cy}, C_c, D_{dc}, and D_{cy} and the static AWC matrices E_1 and E_2 for saturated locally Lipschitz systems such that undesirable saturation effects can be reduced and the predefined H_∞ performance $\|z(t)\| < \gamma \|w(t)\|$ is simultaneously ascertained. Subsequently, we reference some beneficial lemmas that will be used later in the derivation of our results.

Lemma 1 (See the works of Wang et al,[30] Abbaszadeh and Marquez,[31] and Zhang et al[32]).
Consider vectors $\bar{x}(t) \in \Re^n$ *and* $\bar{y}(t) \in \Re^n$, *and for any positive definite matrix* $P \in \Re^{n\times n}$, *the following inequality holds:*

$$2\bar{x}^T(t)\,\bar{y}(t) \le \bar{x}^T(t)\,P\bar{x}(t) + \bar{y}^T(t)\,P^{-1}\bar{y}(t)\,. \tag{15}$$

Lemma 2 (See other works[30-34]).
For vectors $\bar{x}(t), \bar{y}(t) \in \Re^n$, *any positive scalar* ε, *and any real matrices S, D, and F(t) of suitable dimensions, with F(t) satisfying* $I - F^T(t)F(t) \ge 0$, *the following inequality holds:*

$$2\bar{x}^T(t)\,DF(t)\,S\,\bar{y}(t) \le \varepsilon^{-1}\bar{x}^T(t)\,DD^T\bar{x}(t) + \varepsilon\bar{y}(t)\,S^TS\,\bar{y}(t)\,. \tag{16}$$

Lemma 3 (See other works[30-34]).
Presuming any real matrices A, B, C, P, and F(t) of suitable dimensions with $P > 0$ *and F(t) satisfying* $I - F^T(t)F(t) \ge 0$, *if* $P^{-1} - \varepsilon^{-1}BB^T > 0$ *for any positive scalar* ε, *we obtain*

$$(A + BF(t)\,C)^T P\,(A + BF(t)\,C) \le A^T\left(P^{-1} - \varepsilon^{-1}BB^T\right)^{-1}A + \varepsilon C^TC. \tag{17}$$

Lemma 4 (See the work of Abbaszadeh[33] and page 301 in the work of Horn and Johnson[35]).
For any positive definite invertible matrix P, we have

$$\|I_n - P\| = \sigma_{\max}(I_n - P) < 1, \tag{18}$$

where $\sigma_{\max}$ *is the maximum singular value of P.*

3 | AWC SYNTHESIS

To design a regional static AWC, consider an auxiliary region given as

$$S_1(\delta(t), u(t)) = \left\{u(t), \delta(t) \in \Re^m; -\bar{v} \le u(t) - \delta(t) \le \bar{v}\right\}, \tag{19}$$

where

$$\delta(t) = \left(\tilde{C}_c\left(\Theta_f\right) - H\left(\Theta_f\right)\right)\chi(t) + \bar{I}_m\bar{E}\zeta_z(u_n(t)) + \widehat{D}_{ucw}w(t),$$

$H(\Theta_f) = [H_1(\Theta_f)\ H_2(\Theta_f)]$. Here, $\delta(t)$ is an auxiliary vector and $\bar{v}$ is the upper limit on saturation nonlinearity. The dead-zone function satisfies the local sector condition (see the works of Rehan et al[2-4,8] and the references therein). The inequality

$$\zeta_z(u(t))^T W\left[\delta(t) - \zeta_z(u(t))\right] \ge 0 \tag{20}$$

holds true if (19) is fulfilled.

Now, we provide an inequality-based treatment for the given matrices of the robust nonlinear dynamic controller $(A_c, B_{bcf}, B_{cy}, C_c, D_{dcf}, D_{cy})$ and static AWC gain matrices (E_1, E_2), which guarantee the mitigation of saturation effects, robustness against time-varying parametric norm-bounded uncertainties, and asymptotic stability of the closed-loop system.

Theorem 1. *Under Assumptions 1 and 2, consider the uncertain nonlinear systems (1) with actuator saturation and exogenous* $\mathcal{L}_2$ *bounded input. For the given parameters of the robust nonlinear dynamic controller* $(A_c, B_{bcf}, B_{cy}, C_c, D_{dcf}, D_{cy})$ *and static AWC gain matrices* (E_1, E_2), *suppose there exist scalars* $\gamma > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, *and* $\mu > 0$, *matrices* $P_1 = P_1^T > 0 \in \Re^{n\times n}, P_2 = P_2^T > 0 \in \Re^{n\times n}$, $W_1 = W_1^T > 0 \in \Re^{n\times n}$, $W_2 = W_2^T > 0 \in \Re^{n\times n}$, $\Gamma_4 \in \Re^{m\times r}$, $\Gamma_5 \in \Re^{m\times n}$, $\Gamma_6(\Theta) \in \Re^{n\times m}$, *and* $M_2 \in \Re^{m\times m}$, *and a diagonal matrix* $S > 0 \in \Re^{m\times m}$, *such that the following matrix inequalities are fulfilled for all* $\Theta_f \in \mathcal{W}_f$:

$$\sigma_{\max}(I_n - P) < I_n, \tag{21}$$

$$\begin{bmatrix} I_n & \tilde{M}_2 \\ & \varepsilon_2 I \end{bmatrix} > 0, \tag{22}$$

$$\begin{bmatrix} P & \left(\tau_{(h)}0\right)^T \\ * & \mu\kappa_{(h)}^2 \end{bmatrix} \geq 0, \ \textit{for } h = 1, 2, 3, \ldots, n, \tag{23}$$

$$\begin{bmatrix} P & \tilde{C}_c^T\left(\Theta_f\right) - H^T\left(\Theta_f\right) \\ * & \mu\bar{v}_{(k)}^2 \end{bmatrix} \geq 0, \ \textit{for } k = 1, 2, 3, \ldots, m, \tag{24}$$

$$\Xi_1^{(1)} = \begin{bmatrix} \Xi_{11}^{(1)} & \Xi_{12}^{(1)} \\ * & \Xi_{22}^{(1)} \end{bmatrix} < 0, \tag{25}$$

where

$$\Xi_{11}^{(1)} = \begin{bmatrix} \Pi_{11} & \Pi_{12} & P\tilde{B}_w \\ * & \Pi_{22} & WD_{cy}D_{yw} \\ * & * & -I \end{bmatrix},$$

$$\Xi_{12}^{(1)} = \begin{bmatrix} \tilde{C}_z^T\left(\Theta_f\right) & 0 & P\tilde{M}_1 & 0 & 0 & H^T\left(\Theta_f\right)\tilde{M}_3 & \varepsilon_3\tilde{N}^T \\ 0 & 0 & 0 & \tilde{D}_{\zeta_z}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{D}_w^T & 0 & 0 \end{bmatrix},$$

$$\Xi_{22}^{(1)} = \begin{bmatrix} -\frac{1}{3}I & \tilde{M}_2 & 0 & 0 & 0 & 0 & 0 \\ * & -3\varepsilon_1\gamma I & 0 & 0 & 0 & 0 & 0 \\ * & * & -\varepsilon_1\gamma^{-1}I & 0 & 0 & 0 & 0 \\ * & * & * & -\frac{1}{3}\gamma I & 0 & 0 & 0 \\ * & * & * & * & -\frac{1}{3}\gamma I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_3 I & 0 \\ * & * & * & * & * & * & -\varepsilon_3 I \end{bmatrix},$$

$$\Pi_{11} = He\left\{P\tilde{A}\left(\Theta_f\right)\right\} + (\varepsilon_1 + 3\varepsilon_2)\tilde{N}^T\tilde{N},$$
$$\Pi_{12} = P\tilde{B}_{\zeta_z} + H^T\left(\Theta_f\right)W,$$
$$\Pi_{22} = -2W + W\bar{I}_m\overline{E} + \overline{E}^T\bar{I}_m W.$$

Then, the robust nonlinear controller along with the static AWC guarantees the following.

1. *The closed-loop system's state trajectories are asymptotically stable for all initial conditions belonging to the region $\Omega(\mu P)$ if $w(t) = 0$.*
2. *The $\mathcal{L}_2$ gain from $w(t)$ to $z(t)$ is less than γ if $w(t) \neq 0$.*
3. *All the state trajectories of the closed-loop nonlinear system remain within the ellipsoidal region $\chi^T(t)\mu P\chi(t) < 1$, for all $t > 0$.*

Proof. A Lyapunov functional candidate is chosen specified by

$$V(\chi, t) = \chi^T(t)P\chi(t). \tag{26}$$

Consider the objective function

$$J(\chi, t) = \dot{V}(\chi, t) - w^T(t)w(t) + \gamma^{-1}z^T(t)z(t) < 0. \tag{27}$$

Integrating (11) from 0 to T, for $T \to \infty$, yields

$$\int_0^T J(\chi, t)\,dt = V(\chi, T) - V(\chi, 0) - \int_0^T w^T(t)w(t)\,dt + \gamma^{-1}\int_0^T z^T(t)z(t)\,dt < 0. \tag{28}$$

If $\chi(0) = 0$, then we obtain $V(\chi, 0) = 0$. As $V(\chi, T) > 0$, (28) implies that $\|z(t)\|_2^2 < \gamma\|w(t)\|_2^2$. If $\chi(0) \neq 0$, then $V(\chi, 0) \neq 0$, and (28) indicates that $\|z(t)\|_2^2 < \gamma\|w(t)\|_2^2 + \gamma V(\chi, 0)$. The dead-zone nonlinearity satisfies sector condition (20). Therefore, (27) can be written as

$$\begin{aligned}
J(\chi,t) \le\ & 2\chi^T(t)\left(\tilde{A}\left(\Theta_f\right)+\Delta\tilde{A}\left(\Theta_f,t\right)\right)P\chi(t)+\zeta_z(u(t))^T\tilde{B}_{\zeta_z}^T P\chi(t)+w^T(t)\tilde{B}_w^T P\chi(t)\\
&+\zeta_z^T(u_n(t))\tilde{D}_{\zeta_z}^T\gamma^{-1}\tilde{D}_{\zeta_z}\zeta_z(u_n(t))+2\zeta_z^T(u_n(t))\tilde{D}_{\zeta_z}^T\gamma^{-1}\tilde{D}_w w(t)+w^T(t)\tilde{D}_w^T\gamma^{-1}\tilde{D}_w w(t)\\
&+\chi^T(t)P\tilde{B}_{\zeta_z}\zeta_z(u_n(t))+\chi^T(t)P\tilde{B}_w w(t)-w^T(t)w(t)+\chi^T(t)\left(\tilde{C}_z\left(\Theta_f\right)\right.\\
&\left.+\Delta\tilde{C}_z\left(\Theta_f,t\right)\right)^T\gamma^{-1}\left(\tilde{C}_z\left(\Theta_f\right)+\Delta\tilde{C}_z\left(\Theta_f,t\right)\right)\chi(t)+2\chi^T(t)\left(\tilde{C}_z\left(\Theta_f\right)\right.\\
&\left.+\Delta\tilde{C}_z\left(\Theta_f,t\right)\right)\gamma^{-1}\tilde{D}_{\zeta_z}\zeta_z(u_n(t))+2\chi^T(t)\left(\tilde{C}_z\left(\Theta_f\right)+\Delta\tilde{C}_z\left(\Theta_f,t\right)\right)\gamma^{-1}\tilde{D}_w w(t)\\
&+2\zeta_z(u(t))^T W\left[\delta(t)-\zeta_z(u(t))\right]\\
<\ & 0.
\end{aligned} \tag{29}$$

From Equations (2), we can select

$$\Delta\tilde{A}(\Theta_f,t)=\tilde{M}_1\tilde{F}(t)\tilde{N},\ \ \Delta\tilde{C}_z(\Theta_f,t)=\tilde{M}_2\tilde{F}(t)\tilde{N},\ \ \Delta\tilde{C}_c(\Theta_f,t)=\tilde{M}_3\tilde{F}(t)\tilde{N}, \tag{30}$$

$$\begin{aligned}
\tilde{M}_1 &= \begin{bmatrix}\overline{M}_{\Delta A} & \overline{M}_{\Delta B}\end{bmatrix},\ \tilde{F}(t)=\begin{bmatrix}\overline{F}(t) & 0\\ 0 & F(t)\end{bmatrix},\ \tilde{N}=\begin{bmatrix}\overline{N}\\ N\bar{I}_{n\theta}\end{bmatrix},\\
\overline{M}_{\Delta A} &= \begin{bmatrix}M_{pa}+B_{pu}D_{cy}M_{yc} & 0\\ B_{cy}M_{yc} & 0\end{bmatrix},\ \overline{F}(t)=\begin{bmatrix}F(t) & 0\\ 0 & F(t)\end{bmatrix},\\
\overline{N} &= \begin{bmatrix}N & 0\\ 0 & N\end{bmatrix},\ \overline{M}_{\Delta B}=\begin{bmatrix}M_{pf}+B_{pu}D_{cy}M_{yf}\\ B_{cy}M_{yf}\end{bmatrix},\\
\tilde{M}_2 &= \begin{bmatrix}\overline{M}_{\Delta Cz} & \overline{M}_{\Delta Dzf}\end{bmatrix},\ \overline{M}_{\Delta Cz}=\begin{bmatrix}M_{zc}+D_{zu}D_{cy}M_{yc} & 0\end{bmatrix},\\
\overline{M}_{\Delta Dzf} &= (M_{zf}+D_{zu}D_{cy}M_{zf}),\ \ \Delta\tilde{C}_c(\Theta_f,t)=\tilde{M}_3\tilde{F}(t)\tilde{N},\\
\tilde{M}_3 &= \begin{bmatrix}\overline{M}_{\Delta Cy} & \overline{M}_{\Delta Dcy}\end{bmatrix},\ \overline{M}_{\Delta Cy}=\begin{bmatrix}D_{cy}M_{yc} & 0\end{bmatrix},\ \overline{M}_{\Delta Dcy}=D_{cy}M_{yf}.
\end{aligned} \tag{31}$$

By employing Lemmas 1 to 3, we obtain

$$\begin{aligned}
2\chi^T(t)(\tilde{A}(\Theta_f)+\tilde{M}_1\tilde{F}(t)\tilde{N})P\chi(t) \le\ & 2\chi^T(t)\tilde{A}(\Theta_f)P\chi(t)+\varepsilon_1\chi^T(t)\tilde{N}^T\tilde{N}\chi(t)\\
&+\varepsilon_1^{-1}\chi^T(t)P\tilde{M}_1\tilde{M}_1^T P\chi(t),
\end{aligned} \tag{32}$$

$$\left(\tilde{C}_z(\Theta_f)+\tilde{M}_2\tilde{F}(t)\tilde{N}\right)^T\gamma^{-1}\left(\tilde{C}_z(\Theta_f)+\tilde{M}_2\tilde{F}(t)\tilde{N}\right)\le\tilde{C}_z^T(\Theta_f)\left(I-\varepsilon_2^{-1}\tilde{M}_2\tilde{M}_2^T\right)^{-1}\tilde{C}_z(\Theta_f)+\varepsilon_2\tilde{N}^T\tilde{N}, \tag{33}$$

$$\begin{aligned}
2\chi^T(t)\left(\tilde{C}_z(\Theta_f)+\tilde{M}_2\tilde{F}(t)\tilde{N}\right)^T\gamma^{-1}\tilde{D}_{\zeta_z}\zeta_z(u_n(t)) \le\ & \chi^T(t)\left(\tilde{C}_z(\Theta_f)+\tilde{M}_2\tilde{F}(t)\tilde{N}\right)^T\gamma^{-1}\left(\tilde{C}_z(\Theta_f)+\tilde{M}_2\tilde{F}(t)\tilde{N}\right)\chi(t)\\
&+\zeta_z^T(u_n(t))\tilde{D}_{\zeta_z}^T\gamma^{-1}\tilde{D}_{\zeta_z}\zeta_z(u_n(t)),
\end{aligned} \tag{34}$$

$$\begin{aligned}
2\chi^T(t)\left(\tilde{C}_z(\Theta_f)+\tilde{M}_2\tilde{F}(t)\tilde{N}\right)^T\gamma^{-1}\tilde{D}_w w(t) \le\ & \chi^T(t)\left(\tilde{C}_z(\Theta_f)+\tilde{M}_2\tilde{F}(t)\tilde{N}\right)^T\gamma^{-1}\left(\tilde{C}_z(\Theta_f)+\tilde{M}_2\tilde{F}(t)\tilde{N}\right)\chi(t)\\
&+w^T(t)\tilde{D}_w^T\gamma^{-1}\tilde{D}_w w(t),
\end{aligned} \tag{35}$$

$$2\zeta_z^T(u_n(t))\tilde{D}_{\zeta_z}^T\gamma^{-1}\tilde{D}_w w(t)\le\zeta_z^T(u_n(t))\tilde{D}_{\zeta_z}^T\gamma^{-1}\tilde{D}_{\zeta_z}\zeta_z(u_n(t))+w^T(t)\tilde{D}_w^T\gamma^{-1}\tilde{D}_w w(t), \tag{36}$$

$$\begin{aligned}
2\chi^T(t)\left(H(\Theta_f)+\tilde{M}_3\tilde{F}(t)\tilde{N}\right)W\zeta_z(u(t)) \le\ & 2\chi^T(t)H(\Theta_f)W\zeta_z(u(t))+\varepsilon_3\chi^T(t)\tilde{N}^T\tilde{N}\zeta_z(u(t))\\
&+\varepsilon_3^{-1}\chi^T(t)H^T(\Theta_f)\tilde{M}_1\tilde{M}_1^T H(\Theta_f)\zeta_z(u(t)).
\end{aligned} \tag{37}$$

Finally, by using (32)-(36) in (29), we attain

$$\begin{aligned}
J(\chi,t) =\ & 2\chi^T(t)\tilde{A}(\Theta_f)P\chi(t)+(\varepsilon_1+3\varepsilon_2)\chi^T(t)\tilde{N}^T\tilde{N}\chi(t)+\varepsilon_1^{-1}\chi^T(t)P\tilde{M}_1\tilde{M}_1^T P\chi(t)\\
& 3\chi^T(t)\tilde{C}_z^T(\Theta_f)\left(I-\varepsilon_2^{-1}\tilde{M}_2\tilde{M}_2^T\right)^{-1}\tilde{C}_z(\Theta_f)\chi(t)+3\zeta_z^T(u_n(t))\tilde{D}_{\zeta_z}^T\gamma^{-1}\tilde{D}_{\zeta_z}\zeta_z(u_n(t))\\
& 2\chi^T(t)P\tilde{B}_{\zeta_z}\zeta_z(u_n(t))+2\chi^T(t)P\tilde{B}_w w(t)-w^T(t)w(t)\\
&+3w^T(t)\tilde{D}_w^T\gamma^{-1}\tilde{D}_w w(t)+2\zeta_z(u(t))^T W\left[\delta(t)-\zeta_z(u(t))\right]\\
<\ & 0.
\end{aligned} \tag{38}$$

Employing the Schur complement to (38) and using (37) yield

$$J(\chi,t) = \mathfrak{I}^T(t)\Xi_1\mathfrak{I}(t) < 0, \tag{39}$$

$$\mathfrak{I}^T(t) = \begin{bmatrix} \chi(t) & \zeta_z(u(t)) & w(t) \end{bmatrix}, \tag{40}$$

$$\Xi_1 = \begin{bmatrix} \overline{\Pi}_{11} & \overline{\Pi}_{12} & P\tilde{B}_w \\ * & \overline{\Pi}_{22} & WD_{cy}D_{yw} \\ * & * & -I + 3\tilde{D}_w^T\gamma^{-1}\tilde{D}_w \end{bmatrix}, \tag{41}$$

$$\begin{aligned} \overline{\Pi}_{11} &= P\tilde{A}(\Theta_f) + \tilde{A}^T(\Theta_f)P + \varepsilon_1\tilde{N}^T\tilde{N} + \varepsilon_1^{-1}P\tilde{M}_1\tilde{M}_1^TP \\ &\quad + 3\tilde{C}_z^T(\Theta_f)\left(I - \varepsilon_2^{-1}\tilde{M}_2\tilde{M}_2^T\right)^{-1}\tilde{C}_z(\Theta_f) + 3\varepsilon_2\tilde{N}^T\tilde{N}, \\ \overline{\Pi}_{12} &= P\tilde{B}_{\zeta_z} + H^T(\Theta_f)W + \varepsilon_3N^TN + \varepsilon_3^{-1}H^T(\Theta_f)\tilde{M}_3\tilde{M}_3^TH(\Theta_f), \\ \overline{\Pi}_{22} &= -2W + W\overline{I}_m\overline{E} + \overline{E}^T\overline{I}_mW + 3\tilde{D}_{\zeta_z}^T\gamma^{-1}\tilde{D}_{\zeta_z}. \end{aligned} \tag{42}$$

By using the Schur complement to (41), we gain (25). Inequality (21) is necessary to ensure the positive definiteness and inevitability of matrix P. Inequality (22) is obligatory for Lemma 3. Condition (23) is attained by including $\Omega(\mu P) = \chi^T(t)\mu P\chi(t) < 1$ in $S_2(\kappa_h) = \{\chi(t)|\chi(t)|_h \le (\kappa_h\ 0)\}$, for $h = 1, 2, 3, \dots, n$. Similarly, by including $\Omega(\mu P) = \chi^T(t)\mu P\chi(t) < 1$ in $S_1(\delta(t)u(t)) = \{\chi(t)\,\|\tilde{C}_c(\Theta_f) - H(\Theta_f)\|\,\chi(t)_k \le \overline{v}_k\}$, where $k = 1, 2, 3, \dots, m$, we attain the matrix inequality (24). Both (23) and (24) are required in our design approach. Condition (23) is useful for dealing with locally Lipschitz systems rather than the simple form of globally Lipschitz systems. Condition (24) is used to guarantee a local region of stability. □

Remark 1. Many controller design schemes for nonlinear systems have been proposed in the literature.[36-38] However, most of the controller design techniques are based on the assumption that manipulating the control signal operates in the linear region. These controllers may provide significant closed-loop stability and performance in the absence of input saturation. However, a real-world transducer cannot transport an unrestricted energy signal, consequently causing actuator saturation, which leads to performance degradation and instability (see the works of da Silva et al[5] and Ran et al[18]). The core objective of this article is to propose a multiobjective AWC-based dynamic robust nonlinear controller. The dynamic robust nonlinear controller provides robustness against time-varying parametric norm-bounded uncertainties, the asymptotic stability of the closed-loop system under zero external disturbances, and the attenuation of disturbance effects under nonzero external disturbances. Meanwhile, the static AWC guarantees the mitigation of saturation effects by using the difference between the saturated and unsaturated manipulated control signals.

Remark 2. Numerous controller and AWC design schemes for nonlinear systems are presented in the literature.[2-5,7,8,36] However, a generalized technique for the simultaneous design of a robust nonlinear controller and static AWC for uncertain nonlinear systems under actuator saturation and exogenous $\mathcal{L}_2$ bounded input has not been completely addressed. In comparison with the conventional design approaches, there are many attributes of the anticipated multiobjective AWC-based dynamic robust nonlinear controller. For example, the present design approach is based on the LPV reformulation property of Lipschitz nonlinear functions compared with the works of Rehan et al.[2-4] Furthermore, parametric uncertainties are considered in the present work to design a robust nonlinear controller and AWC. Moreover, nonlinearities have been considered in the state in addition to the output equations of a plant. In addition, external perturbations have been considered in the state along with the output equations of a plant.

Remark 3. The proposed AWC-based robust nonlinear controller synthesis methodology possesses unique features in contrast to the existing techniques of AWC design[2-4,8] due to the application of the generalized and less conservative Lipschitz nonlinearity condition. In contrast to the conventional Lipschitz schemes,[2-4] we used the LPV-based reformulation Lipschitz technique[7,29] to design a robust nonlinear controller and static AWC for uncertain nonlinear systems. In comparison with the conventional Lipschitz condition, the reformulated Lipschitz condition can be used to synthesize a less conservative controller and AWC for nonlinear control systems. The LPV technique permits the effective treatment of the nonlinear function with large Lipschitz parameters by employing the nonlinear part of the plant for different state values. The conventional Lipschitz properties[2,3,8] do not characterize the unique characteristics of nonlinear dynamics, whereas the reformulation Lipschitz condition reveals the true representation of the nonlinear function and is the premium one that contains all the characteristics of the nonlinear dynamics. The reformulation Lipschitz property has not been hitherto employed for the simultaneous design of a nonlinear controller and static AWC for uncertain nonlinear systems.

The results of Theorem 1 can be reduced to the case of the nonlinear system with $\Delta\tilde{A}(\Theta_f, t) = 0$, $\Delta\tilde{C}_z(\Theta_f, t) = 0$, and $\tilde{C}_c(\Theta_f, \Delta, t) = 0$ in (1), as observed in the work of Wang et al.[7] The corresponding result is provided as follows.

Corollary 1. *Under Assumptions 1 and 2, consider the uncertain nonlinear systems (1) with $\Delta\tilde{A}(\Theta_f, t) = 0$, $\Delta\tilde{C}_z(\Theta_f, t) = 0$, and $\tilde{C}_c(\Theta_f, \Delta, t) = 0$ under actuator saturation and exogenous $\mathcal{L}_2$ bounded inputs. The robust nonlinear controller along with the static AWC can be designed if there exist a scalar, $\gamma > 0$, matrices $P_1 = P_1^T > 0 \in \mathfrak{R}^{n\times n}$, $P_2 = P_2^T > 0 \in \mathfrak{R}^{n\times n}$, $W_1 = W_1^T > 0 \in \mathfrak{R}^{n\times n}$, $W_2 = W_2^T > 0 \in \mathfrak{R}^{n\times n}$, $\Gamma_1 \in \mathfrak{R}^{n\times n}$, $\Gamma_2 \in \mathfrak{R}^{n\times q}$, $\Gamma_3 \in \mathfrak{R}^{n\times r}$, $\Gamma_4 \in \mathfrak{R}^{m\times r}$, $\Gamma_5 \in \mathfrak{R}^{m\times n}$, $\Gamma_6(\Theta) \in \mathfrak{R}^{n\times m}$, $\Gamma_7 \in \mathfrak{R}^{n\times m}$, $D_{cy} \in \mathfrak{R}^{n\times n}$, and $M_2 \in \mathfrak{R}^{m\times m}$, and a diagonal matrix $S > 0 \in \mathfrak{R}^{m\times m}$, such that conditions (21)-(24) and the following matrix inequality are fulfilled for all $\Theta_f \in \mathcal{W}_f$:*

$$\Xi_1^{*1} = \begin{bmatrix} \overline{\Pi}_{11}^{*1} & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} \\ * & \overline{\Pi}_{22}^{*1} & \psi_{23} & \psi_{24} & \psi_{25} \\ * & * & \psi_{33} & D_{cy}D_{yw} & \overline{\Pi}_{35}^{*1T} \\ * & * & * & -I & \overline{\Pi}_{45}^{*1T} \\ * & * & * & * & -\gamma I \end{bmatrix} < 0, \tag{43}$$

$$\begin{aligned} \overline{\Pi}_{11}^{*1} &= He\left\{A_pW_1 + B_u\Gamma_5 + B_{pf}\Theta I_nW_1 + B_u\Gamma_4\Theta I_nW_1\right\}, \\ \overline{\Pi}_{22}^{*1} &= He\left\{P_1A_p + \Gamma_2C_y + \Gamma_3\Theta I_n\right\},\ \overline{\Pi}_{35}^{*1} = (D_{zu}Q_2 - D_{zu}S), \\ \overline{\Pi}_{45}^{*1} &= \left(D_{zw} + D_{zu}D_{cy}D_{yw}\right). \end{aligned} \tag{44}$$

Then, the robust nonlinear controller along with the static AWC (4) guarantees the following.

1. *The closed-loop system state trajectories are asymptotically stable for all initial conditions belonging to the region $\Omega(\mu P)$ if $w(t) = 0$.*
2. *The $\mathcal{L}_2$ gain from $w(t)$ to $z(t)$ is less than γ if $w(t) \neq 0$.*
3. *All the state trajectories of the closed-loop system remain bounded within the ellipsoidal region $\chi(t)^T\mu P\chi(t) < 1$, for all $t > 0$.*

Remark 4. The suggested approach in Theorem 1 considers the conventional schemes[7] as a specific case by ignoring parametric uncertainties. It is notable that the effect of parametric uncertainties has been rarely addressed in the AWC or AWC-based controller design of nonlinear systems. The present approach in Theorem 1 considers parametric uncertainties in both the linear and nonlinear components of the plant. The consideration of parametric uncertainties in both components, for designing several controller and AWC gains, is a nontrivial research problem. Moreover, AWC-based control for locally Lipschitz nonlinear systems has not been thoroughly considered in the previous studies, as observed in the work of Wang et al.[7]

Now, we present a BMI-based methodology for determining the gains of robust nonlinear dynamic controller matrixes $(A_c, B_{bcf}, B_{cy}, C_c, D_{dcf}, D_{cy})$ and static AWC gain matrices (E_1, E_2).

Theorem 2. *Under Assumptions 1 and 2, consider the uncertain nonlinear systems (1) with actuator saturation nonlinearity and exogenous $\mathcal{L}_2$ bounded inputs. The robust nonlinear controller along with the static AWC can be designed if there exist scalars $\gamma > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$, and $\mu > 0$, matrices $P_1 = P_1^T > 0 \in \mathfrak{R}^{n\times n}$, $P_2 = P_2^T > 0 \in \mathfrak{R}^{n\times n}$, $W_1 = W_1^T > 0 \in \mathfrak{R}^{n\times n}$, $W_2 = W_2^T > 0 \in \mathfrak{R}^{n\times n}$, $\Gamma_1 \in \mathfrak{R}^{n\times n}$, $\Gamma_2 \in \mathfrak{R}^{n\times q}$, $\Gamma_3 \in \mathfrak{R}^{n\times r}$, $\Gamma_4 \in \mathfrak{R}^{m\times r}$, $\Gamma_5 \in \mathfrak{R}^{m\times n}$, $\Gamma_6(\Theta) \in \mathfrak{R}^{n\times m}$, $\Gamma_7 \in \mathfrak{R}^{n\times m}$, $D_{cy} \in \mathfrak{R}^{m\times q}$, and $Q_2 \in \mathfrak{R}^{m\times m}$, and a diagonal matrix $S > 0 \in \mathfrak{R}^{m\times m}$, such that the following matrix inequalities are fulfilled for all $\Theta_f \in \mathcal{W}_f$:*

$$\begin{bmatrix} I_n & I_n - P^T \\ * & I_n \end{bmatrix} > 0, \tag{45}$$

$$\begin{bmatrix} W_1 & I & \tau_{(h)}W_1^T \\ * & P_1 & \tau_{(h)}I \\ * & * & \mu\kappa_{(h)}^2 \end{bmatrix} \geq 0,\ \textit{for}\ h = 1, 2, 3, \ldots, n, \tag{46}$$

$$\begin{bmatrix} W_1 & I & \psi^{*T}_{13(k)} \\ * & P_1 & \psi^{*T}_{23(k)} \\ * & * & \mu\bar{v}^{2}_{(k)} \end{bmatrix} \geq 0, \ for\ k = 1, 2, 3, \ldots, m, \tag{47}$$

$$\begin{bmatrix} \varepsilon_2 I & M_{zc} + D_{zu}D_{cy}M_{cy} & 0 & M_{zf} + D_{zu}D_{cy}M_{zf} \\ * & I & 0 & 0 \\ * & * & I & 0 \\ * & * & * & I \end{bmatrix} > 0, \tag{48}$$

$$\Xi^{(2)}_1 = \begin{bmatrix} \Xi^{(2)}_{11} & \Xi^{(2)}_{12} \\ * & \Xi^{(2)}_{22} \end{bmatrix} < 0, \tag{49}$$

$$\Xi^{(2)}_{11} = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} & 0 & 0 & 0 \\ * & \psi_{22} & \psi_{23} & \psi_{24} & \psi_{25} & 0 & 0 & 0 \\ * & * & \psi_{33} & D_{cy}D_{yw} & 0 & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & * & \gamma\psi_{55} & \psi_{56} & 0 & \psi_{58} \\ * & * & * & * & * & 3\gamma^{-1}\psi_{99} & 0 & 0 \\ * & * & * & * & * & * & 3\gamma^{-1}\psi_{99} & 0 \\ * & * & * & * & * & * & * & 3\gamma^{-1}\psi_{99} \end{bmatrix},$$

$$\Xi^{(2)}_{12} = \begin{bmatrix} \psi_{19} & 0 & \psi_{1(11)} & 0 & 0 & \psi_{1(14)} & 0 & \psi_{1(16)} & \varepsilon_3 N^T & 0 & \varepsilon_3(N\Theta I)^T \\ \psi_{29} & 0 & \psi_{2(11)} & 0 & 0 & \psi_{2(14)} & 0 & \psi_{2(16)} & 0 & \varepsilon_3 N^T & 0 \\ 0 & 0 & 0 & \psi^T_{3(12)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi^T_{4(13)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Xi^{(2)}_{22} == \text{diag}\left\{\psi_{99}, \psi_{99}, \psi_{99}, \gamma\psi_{55}, \gamma\psi_{55}, -\varepsilon_3 I, -\varepsilon_3 I, -\varepsilon_3 I, -\varepsilon_3 I, -\varepsilon_3 I, -\varepsilon_3 I\right\},$$

$$\begin{aligned}
\psi^*_{13} &= \Gamma_4\Theta W_1 + \Gamma_5 - \Gamma^T_6, \ \psi^*_{23} = D_{cy}C_y + \Gamma_4\Theta - H_1, \\
\psi_{11} &= He\left\{A_pW_1 + B_u\Gamma_5 + B_{pf}\Theta I_n W_1 + B_u\Gamma_4\Theta I_n W_1\right\} \\
&\quad + (\varepsilon_1 + 3\varepsilon_2)(N^TN + (N\Theta I)^T(N\Theta I)), \\
\psi_{12} &= A_p + B_uD_{cy}C_y + \Gamma^T_1 + B_{pf}\Theta + B_u\Gamma_4\Theta + (\Gamma_3\Theta W_1)^T, \\
\psi_{13} &= B_uQ_2 - B_uS + \Gamma_6, \ \psi_{14} = B_w + B_uD_{cy}D_{yw}, \\
\psi_{15} &= (C_zW_1 + D_{zu}\Gamma_5)^T + W_1\Theta^T(D_{zf} + D_{zu}\Gamma_4)^T, \\
\psi_{19} &= P^T_1M_{pA} + P^T_1B_{pu}C_{cy}M_{yc} + P_2B_{cy}M_{yc}, \\
\psi_{1(11)} &= P^T_1M_{pf} + P^T_1B_{pu}D_{cy}M_{yf} + P_2B_{cy}M_{yf}, \\
\psi_{1(14)} &= H^T_{1k}(\Theta)D_{cy}M_{yc}, \ \psi_{2(14)} = H^T_{2k}(\Theta)D_{cy}M_{yc}, \\
\psi_{1(16)} &= H^T_{1k}(\Theta)D_{cy}M_{yf}, \ \psi_{2(16)} = H^T_{2k}(\Theta)D_{cy}M_{yf}, \\
\psi_{22} &= He\left\{P_1A_p + \Gamma_2C_y + \Gamma_3\Theta I_n\right\} + (\varepsilon_1 + 3\varepsilon_2)(N^TN), \\
\psi_{23} &= \Gamma_7 + H^T_1(\Theta), \ \psi_{24} = P_1B_w + \Gamma_2D_{yw}, \\
\psi_{25} &= (C_z + D_{zu}D_{cy}C_y)^T + \Theta(D_{zf} + D_{zu}\Gamma_4)^T, \\
\psi_{29} &= P^T_2M_{pA} + P^T_2B_{pu}C_{cy}M_{yc} + P_1B_{cy}M_{yc}, \\
\psi_{2(11)} &= P^T_2M_{pf} + P^T_2B_{pu}D_{cy}M_{yf} + P_1B_{cy}M_{yf}, \\
\psi_{33} &= -2S + He\left\{Q_2\right\}, \ \psi_{3(12)} = (D_{zu}Q_2 - D_{zu}S), \\
\psi_{3(13)} &= (D_{zw} + D_{zu}D_{cy}D_{yw}), \ \psi_{4(13)} = (D_{zw} + D_{zu}D_{cy}D_{yw})^T, \\
\psi_{56} &= M_{zc} + D_{zu}D_{cy}M_{cy}, \ \psi_{58} = M_{zf} + D_{zu}D_{cy}M_{zf}, \\
\psi_{55} &= -\frac{1}{3}I, \ \psi_{99} = -\varepsilon_1 I.
\end{aligned} \tag{50}$$

Then, the robust nonlinear controller along with the static AWC (4) guarantees the following.

(1) *The closed-loop system state trajectories are asymptotically stable for all initial conditions belonging to the region* $\Omega(\mu P)$ *if* $w(t) = 0$.
(2) *The* $\mathcal{L}_2$ *gain from* $w(t)$ *to* $z(t)$ *is less than* γ *if* $w(t) \neq 0$.
(3) *All the state trajectories of the closed-loop system remain bounded within the ellipsoidal region* $\chi^T(t)\mu P\chi(t) < 1$, *for all* $t > 0$.

Moreover, the robust nonlinear controller and the static AWC gain matrices can be constructed by

$$\begin{aligned} A_c &= P_2^{-1}\left(\Gamma_1 - P_1A_pW_1 - P_1B_u\Gamma_5 - P_2B_{cy}C_yW_1\right)W_2^{-T}, \\ B_{bcf} &= P_2^{-1}\left(\Gamma_3 - P_1B_{pf} - \Gamma_2D_{yf} - P_1B_uD_{dcf}\right), E_2 = Q_2S^{-1}, \\ B_{cy} &= P_2^{-1}\left(\Gamma_2 - P_1B_uD_{cy}\right),\ C_c = \left(\Gamma_5 - D_{cy}C_yW_1\right)W_2^{-T}, \\ D_{dcf} &= \left(\Gamma_4 - D_{cy}D_{yf}\right),\ E_1 = P_2^{-1}\left(\Gamma_7 - P_1B_uQ_2 + P_1B_uS\right)S^{-1}. \end{aligned} \tag{51}$$

Proof. The following matrices are selected:

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_1^T \end{bmatrix},\ \aleph = \begin{bmatrix} W_1 & W_2 \\ W_2^T & W_1^T \end{bmatrix},\ \text{and } \overline{\aleph} = \begin{bmatrix} W_1 & W_2 \\ I_n & 0 \end{bmatrix}. \tag{52}$$

By pre- and post-multiplying inequality (25) in Theorem 1 by the diagonal matrix $\mathrm{diag}(\overline{\aleph}, S, I, I, I, I, I, I, I, I)$ and using the change in variables

$$\begin{aligned} \Gamma_1 &= \left(P_1A_pW_1 + P_1B_u\Gamma_5 + P_2B_{cy}C_yW_1 + P_2A_cW_2^T\right), \\ \Gamma_2 &= \left(P_1B_uD_{cy} + P_2B_{cy}\right), \\ \Gamma_3 &= \left(P_1B_{pf} + \Gamma_2D_{yf} + P_1B_uD_{dcf} + P_2B_{bcf}\right), \\ \Gamma_4 &= \left(D_{cy}D_{yf} + D_{dcf}\right), \\ \Gamma_5 &= \left(D_{cy}C_yW_1 + C_cW_2^T\right), \\ \Gamma_6 &= \left(W_1H_1(\Theta)^T + W_2H_2(\Theta)^T\right), \\ \Gamma_7 &= \left(P_1B_uQ_2 - P_1B_uS + P_2I_nQ_1\right), \\ W^{-1} &= S, Q_1 = E_1S, Q_2 = E_2S, \end{aligned} \tag{53}$$

we obtain the BMI (49). The LMIs (45), (46), and (47) are obtained by employing the Schur complement to (21), (23), and (24), respectively. The LMI (48) is attained from (22). This completes the proof of Theorem 1. □

Remark 5. In comparison to Theorem 1, Theorem 2 provides a BMI-based methodology for determining the values of robust nonlinear dynamic controller matrices $(A_c, B_{bcf}, B_{cy}, C_c, D_{dcf}, D_{cy})$ and static AWC gain matrices (E_1, E_2). It may be challenging to design the controller and AWC from the constraints of Theorem 1 because the tuning efforts are obligatory for finding appropriate controller and AWC gains. However, the inequalities in Theorem 2 can be solved directly for the calculation controller and AWC gains by employing the ILMI algorithm, convex optimization procedure, $\mathcal{L}_2$ gain minimization, and cone complementary linearization technique.

Remark 6. Conditions (46) and (47) indicate that $(I - P_1W_1) > 0$. As inequality (45) guarantees the nonsingularity of the matrix P, there always exist nonsingular matrices P_2 and W_2 such that $P_2W_2^T = (I - P_1W_1)$. Therefore, for the sake of simplicity, one can select any nonsingular value for matrix W_2, such as $W_2 = I_n$, and the matrix P_2 can then be calculated as $P_2 = (I - P_1W_1)(W_2^T)^{-1}$. By employing a change in the variable as $P_2 = X_2$, $\overline{P}_2 = \overline{X}_2$, $\overline{P}_2 = P_2^{-1}$, and $\overline{X}_2 = X_2^{-1}$, the inequalities in Theorem 2 can be solved via convex routines with the following objective function and additional constraints:

$$\mathrm{trace}\left(P_2\overline{X}_2 + \overline{P}_2X_2 + X_2\overline{X}_2 + P_{20}\overline{X}_2 + P_2\overline{X}_{20} + \overline{P}_{20}X_2 + \overline{P}_2X_{20} + X_2\overline{X}_{20}\right) \tag{54}$$

subject to

$$\begin{bmatrix} X_2 & I \\ I & \overline{X}_2 \end{bmatrix} \geq 0, \begin{bmatrix} X_2 & I \\ I & \overline{P}_2 \end{bmatrix} \geq 0. \tag{55}$$

Remark 7. The constraints in Theorem 2 are expressed in BMI form due to the presence of the nonlinear terms $\Gamma_3\Theta I_n W_1$ and $\Gamma_4\Theta I_n W_1$. Two methods can be used to obtain the optimal solution. First, the ILMI optimization technique proposed in the work of Peaucelle and Arzelier[39] can be used to determine the values of robust nonlinear dynamic controller matrices and static AWC gains. Secondly, the BMI condition in Theorem 2 can be converted to an LMI by taking $\Gamma_3 = 0$ and $\Gamma_4 = 0$. However, under this condition, conservatism is introduced in the condition of Theorem 2, which cannot be ignored to obtain LMIs.

By using the approaches suggested in Remarks 6 and 7, the constraints in Theorem 2 can be resolved using convex routines through the iterative solution of LMIs and optimization of the nonlinear objective function. In Corollary 2, we derive a BMI-based global technique for the simultaneous design of a robust nonlinear controller and AWC for uncertain nonlinear systems under actuator saturation and exogenous $\mathcal{L}_2$ bounded disturbances by employing the global sector condition.[2]

Corollary 2. *Under Assumptions 1 and 2, consider the uncertain nonlinear systems (1) with actuator saturation and exogenous $\mathcal{L}_2$ bounded inputs. The robust nonlinear controller along with the static AWC can be designed if there exist scalars $\gamma > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\mu > 0$, matrices $P_1 = P_1^T > 0 \in \Re^{n\times n}$, $P_2 = P_2^T > 0 \in \Re^{n\times n}$, $W_1 = W_1^T > 0 \in \Re^{n\times n}$, $W_2 = W_2^T > 0 \in \Re^{n\times n}$, $\Gamma_1 \in \Re^{n\times n}$, $\Gamma_2 \in \Re^{n\times q}$, $\Gamma_3 \in \Re^{n\times r}$, $\Gamma_4 \in \Re^{m\times r}$, $\Gamma_5 \in \Re^{m\times n}$, $\Gamma_6(\Theta) \in \Re^{n\times m}$, $\Gamma_7 \in \Re^{n\times m}$, $D_{cy} \in \Re^{n\times n}$, and $M_2 \in \Re^{m\times m}$, and a diagonal matrix $S > 0 \in \Re^{m\times m}$, such that the following matrix inequality is fulfilled for all $\Theta_f \in \mathcal{W}_f$:*

$$\Xi_1^{*2} = \begin{bmatrix} \Xi_{11}^{*2} & \Xi_{12}^{(2)} \\ * & \Xi_{22}^{(2)} \end{bmatrix} < 0,$$

$$\Xi_{11}^{*2} = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13}^{*2} & \psi_{14} & \psi_{15} & 0 & 0 & 0 \\ * & \psi_{22} & \psi_{23}^{*2} & \psi_{24} & \psi_{25} & 0 & 0 & 0 \\ * & * & \psi_{33} & D_{cy}D_{yw} & 0 & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & * & \gamma\psi_{55} & \psi_{56} & 0 & \psi_{58} \\ * & * & * & * & * & 3\gamma^{-1}\psi_{99} & 0 & 0 \\ * & * & * & * & * & * & 3\gamma^{-1}\psi_{99} & 0 \\ * & * & * & * & * & * & * & 3\gamma^{-1}\psi_{99} \end{bmatrix}, \tag{56}$$

$$\psi_{13}^{*2} = B_uQ_2 - B_uS + W_1\left(D_{cy}C_y + \Gamma_4\Theta I\right)^T + W_2C_c^T,$$

$$\psi_{23}^{*2} = \Gamma_7 + \left(D_{cy}C_y + \Gamma_4\Theta I\right)^T.$$

Then, the robust nonlinear dynamic controller along with the static AWC (4) guarantees the following.

(1) The closed-loop system state trajectories are asymptotically stable for all initial conditions belonging to the region $\Omega(\mu P)$ if $w(t) = 0$.
(2) The $\mathcal{L}_2$ gain from $w(t)$ to $z(t)$ is less than γ if $w(t) \neq 0$.

Remark 8. Theorem 2 provides an inequality-based local static AWC-based robust nonlinear dynamic controller design schema. The constraints in Theorem 1 can be employed to design the robust controller and AWC for stable and unstable uncertain nonlinear systems. If a nonlinear system is globally asymptotically stable and satisfies the globally Lipschitz condition, the novel condition provided in Corollary 2 can be used for the simultaneous design of a robust nonlinear controller and AWC for uncertain nonlinear systems under actuator saturation and exogenous $\mathcal{L}_2$ bounded disturbances. A similar condition for the globally Lipschitz nonlinear systems can also be derived from Theorem 2 as a special case.

4 | SIMULATION RESULTS

Two simulation examples are provided in this section to demonstrate the effectiveness of the proposed AWC-based dynamic robust nonlinear controller design schemes.

Example 1. Consider the nonlinear system[18] given by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -\frac{1}{24}(x_1(t) - x_2(t)) + \frac{1}{12}\left(\sin\left(\frac{x_1}{2}\right)\cos\left(\frac{x_3}{3}\right) + \cos\left(\frac{x_1}{3}\right)\sin\left(\frac{x_2}{2}\right)\right) + 3\mathcal{N}_{sat}(u_c(t)), \\ y(t) &= x_1(t). \end{aligned} \tag{57}$$

The saturation level is taken as $\overline{v} = \pm 1$. For $\kappa_f = 0.0833$, the nonlinear dynamics satisfies the Lipschitz condition globally. Our goal is to compute the static AWC gain matrices E_1 and E_2 to achieve a region of stability, $x(t)^T \mu P_1 x(x) < 1$, for all $t > 0$ that is as large as possible. By solving Theorem 2 for the optimization problem, we attain

$$P_1 = \begin{bmatrix} 0.0024 & -0.0022 \\ -0.0022 & 0.0026 \end{bmatrix}, \; E_1 = \begin{bmatrix} 0.0027 \\ 0.0026 \end{bmatrix}, \; E_2 = 0.5, \; \gamma = 0.167. \tag{58}$$

An important objective of the simultaneous design of the robust nonlinear controller and AWC is the enlargement of the region of attraction (ROA) of the closed-loop system. To achieve a large estimate of the ROA, we need to minimize trace (P_1). For this purpose, we considered the subsequent optimization problem[5] given by

$$\min \operatorname{trace}(H_1), \begin{bmatrix} H_1 & I \\ I & Q_1 \end{bmatrix} \geq 0, \; (45)-(49), \tag{59}$$

where $H_1 \geq Q_1^{-1} = P_1$ is a positive definite matrix. The minimization of trace(H_1) leads to the minimization of trace(P_1) and, consequently, an implicit maximization of the ellipsoidal region.

A comparative analysis of the ROA of the existing techniques, the active disturbance rejection control–based AWC,[18] sector-based AWC,[19] static AWC methodology,[5] and proposed windup compensation-based control technique is shown in Table 1. From the matrices P_1 and their respective traces trace(P_1), we can conclude that, in contrast to the existing techniques, a significantly large estimate of the ellipsoidal basin of attraction is obtained with the proposed technique in Theorem 2. The $\mathcal{L}_2$ gain minimization from $w(t)$ to $z(t)$ can be considered by solving the constraints (45)-(49) of Theorem 2 for the optimization of γ. The optimal values of the $\mathcal{L}_2$ gain γ obtained from different techniques are shown in Table 2. For comparison, estimates of the region of stability are shown in Figure 1, where R_1 is the estimated ROA by Theorem 2, R_2 is the estimated ROA by da Silva et al,[5] R_3 is the estimated ROA by da Silva et al,[19] and R_4 is the estimated ROA by Ran et al.[18] In contrast to the existing approaches, the technique proposed in Theorem 2 has the largest estimated ROA.

TABLE 1 Comparisons of the estimated basin of attraction for different techniques for Example 1

Techniques	P_1	trace (P_1)
Ran et al[18]	$\begin{bmatrix} 1.1554 & 0.3387 \\ 0.3387 & 0.5999 \end{bmatrix}$	1.7553
da Silva et al[19]	$\begin{bmatrix} 0.2591 & -0.0039 \\ -0.0039 & 0.0207 \end{bmatrix}$	0.2798
da Silva et al[5]	$\begin{bmatrix} 13.8747 & -7.3119 \\ -7.3119 & 339.5162 \end{bmatrix}$	353.3909
Proposed method of Theorem 2	$\begin{bmatrix} 0.0024 & -0.0022 \\ -0.0022 & 0.0026 \end{bmatrix}$	0.0050

TABLE 2 Comparisons of the performances for different techniques for Example 1

$\mathcal{L}_2$ Gain Minimization	Saqib et al[23]	Ran et al[18]	Theorem 2
γ	0.190×10^4	0.6527	0.1670

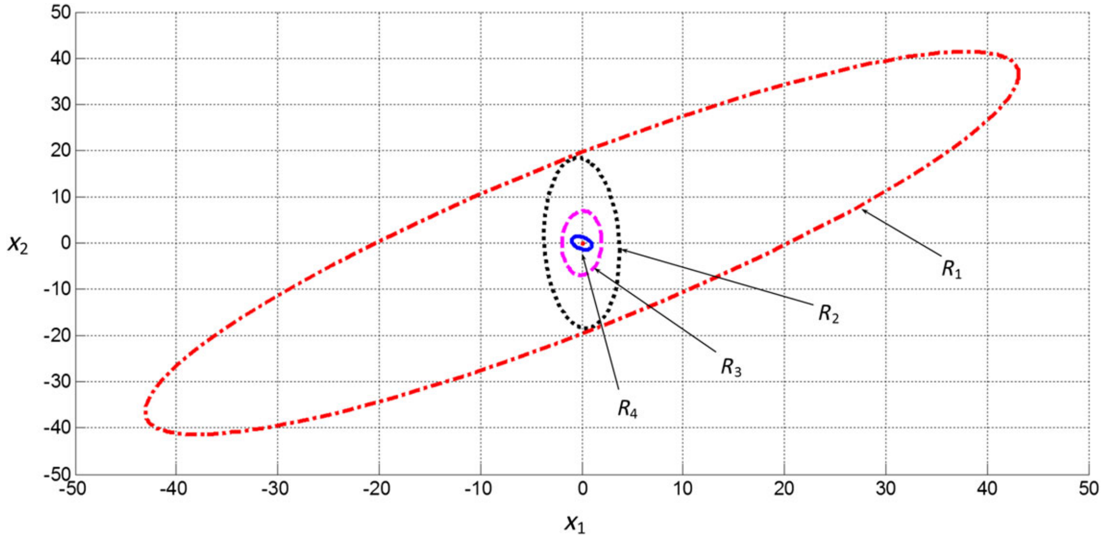


FIGURE 1 Comparison of estimated region of attraction of closed-loop nonlinear system [Colour figure can be viewed at wileyonlinelibrary.com]

Example 2. Consider the following nonlinear one-link flexible robot[40]:

$$\begin{aligned}
\dot{x}_m(t) &= x_{wm}(t), \\
\dot{x}_{wm}(t) &= \frac{k}{J_m}x_L(t) - \frac{k}{J_m}x_m(t) - \frac{B}{J_m}x_{wm}(t) + \frac{K_a}{J_m}\mathcal{N}_{sat}(u(t)), \\
\dot{x}_L(t) &= x_{wL}(t), \\
\dot{x}_{wL}(t) &= -\frac{k}{J_L}x_L(t) + \frac{k}{J_L}x_m(t) - \frac{MgL}{J_L}\sin(x_L(t)),
\end{aligned} \tag{60}$$

where $x_m(t)$ is the angular position of the motor rotor, $x_L(t)$ represents the angular position of the link, $\dot{x}_m(t)$ symbolizes the angular velocity of the motor rotor, and $\dot{x}_L(t)$ is the angular velocity of the link. J_m denotes the inertia of the motor rotor, and J_L signifies the inertia of the link. The details of the parameters (k, B, K_a, M, g, and $2L$) are specified in Table 3. The saturated controlled torque to the shaft is represented by $\mathcal{N}_{sat}(u(t))$. $\frac{MgL}{J_L}\sin(x_L(t))$ characterizes the flexible joint nonlinearity. The position of the motor rotor $x_m(t)$ and angular velocity of link $x_{wL}(t)$ are considered as outputs. Using the values of the parameters itemized in Table 1, the plant (57) can be characterized by

$$\begin{aligned}
A_p &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.64 & -12.43 & 48.68 & 0 \\ 0 & 0 & 0 & 1 \\ 19.35 & 0 & -19.35 & 0 \end{bmatrix}, \; B_{pf} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4.536 \end{bmatrix}, \; B_{pu} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \\
f(t,x) &= \sin(x_L(t)), \; C_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; C_z = -\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{61}$$

TABLE 3 Nonlinear one-link flexible robot parameters

Notation	Parameters	Value	Units
J_m	Motor inertia	0.0239	kgm^2
J_L	Link inertia	5.650	kgm^2
B	Viscous friction coefficient	5.650	NmV^{-1}
k	Torsional spring constant	0.0029	$Nm(rad)^{-1}$
K_a	Amplifier gain	5.650	NmV^{-1}
M	Payload mass	5.650	kg
$2L$	Link length	5.650	m
g	Gravity acceleration	9.800	ms^{-2}

It is significant to note that, due to wear-and-tear, high-frequency components, unmodeled dynamics, and environmental changes, no real physical systems can be accurately modeled by mathematical equations. Therefore, a system always contains model uncertainties.[41] The uncertainty matrices are selected as

$$M_{pa} = \begin{bmatrix} 0 & 0.12 & 0 & 0 \\ -0.51 & 0.20 & 0.81 & 0 \\ 0 & 0 & 0 & 0.21 \\ 0.35 & 0 & -.53 & 0 \end{bmatrix}, \; M_{yc} = \begin{bmatrix} 0.12 & 0 & 0 & 0.1 \end{bmatrix},$$
$$M_{zc} = \begin{bmatrix} 0.12 & 0 & 0 & 0.1 \end{bmatrix}, \; M_{pf} = \begin{bmatrix} 0 & 0 & 0 & 0.21 \end{bmatrix}^T, \text{ and } N = 1. \tag{62}$$

The input and output disturbances are selected as

$$d_{p1}(t) = \sin 100t, d_{p2}(t) = \begin{bmatrix} 0.03 \sin 9t & -0.02 \sin 11t \end{bmatrix}^T. \tag{63}$$

The system matrices can be accounted for to solve the robust nonlinear controller and static AWC design. Note that the nonlinear dynamics is globally Lipschitz, and the conventional approaches, such as those by Wang et al[7] and Rehan et al,[8] fail to design a multiobjective controller for this case due to the presence of parametric uncertainties. To design the proposed robust nonlinear controller with a static AWC (4), the controller matrices A_c, B_{bcf}, B_{cy}, C_c, D_{dcf}, and D_{cy} and the static AWC matrices E_1 and E_2 are obtained by solving the constraints (45)-(49) of Theorem 2 using convex routines. Saturation nonlinearity is ubiquitous in all control systems. In engineering systems, all physical actuators are subject to saturation restriction because of its minimum and maximum bounds. The saturation limit $\bar{\upsilon} = \pm 1$ is taken. By taking $\kappa_f = 0.333$, the following controller and static AWC gain matrices are obtained:

$$A_c = 10^2 \times \begin{bmatrix} -17.04 & 90.64 & 1.11 & -14.29 \\ -3.68 & 19.09 & 0.152 & -2.85 \\ 13.87 & -74.79 & -1.11 & 12.09 \\ -1.56 & 7.89 & 0.020 & -1.14 \end{bmatrix}, \; B_{bcf} = \begin{bmatrix} 113.5517 \\ 25.2488 \\ -141.0389 \\ -10.4689 \end{bmatrix},$$
$$B_{cy} = \begin{bmatrix} 2.97 & -17.97 \\ 0.72 & -3.77 \\ -2.25 & 14.86 \\ 0.34 & -1.56 \end{bmatrix}, \; E_1 = \begin{bmatrix} -0.0316 \\ -0.0085 \\ -0.0239 \\ -0.0037 \end{bmatrix}, \; D_{dcf} = [0], \; E_2 = 0.5,$$
$$C_c = \begin{bmatrix} -76.98 & 122.71 & -53.97 & 55.51 \end{bmatrix}, \; D_{cy} = \begin{bmatrix} 59.18 & -13.31 \end{bmatrix}.$$

The output response of the closed-loop uncertain nonlinear one-link flexible robotic system with the initial condition $\begin{bmatrix} x_m(0) & x_{wm}(0) & x_L(0) & x_{wL}(0) \end{bmatrix}^T = \begin{bmatrix} 0.5 & 1 & 3 & 0.8 \end{bmatrix}^T$ is revealed in Figure 2. The closed-loop system response

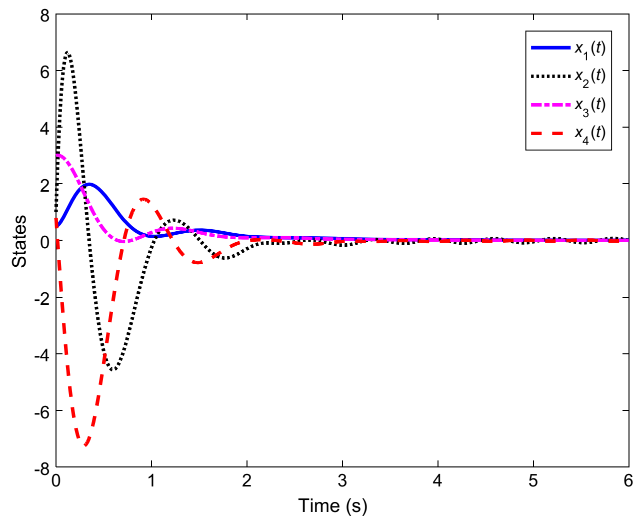


FIGURE 2 Convergence of trajectories of closed-loop uncertain nonlinear one-link flexible robotic system using the proposed method [Colour figure can be viewed at wileyonlinelibrary.com]

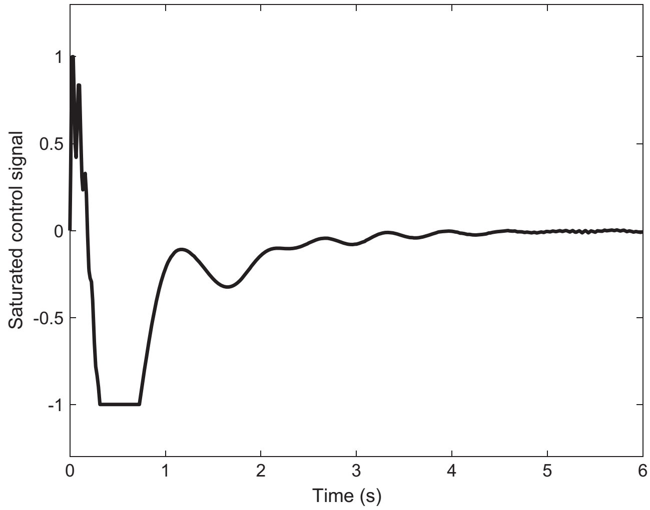
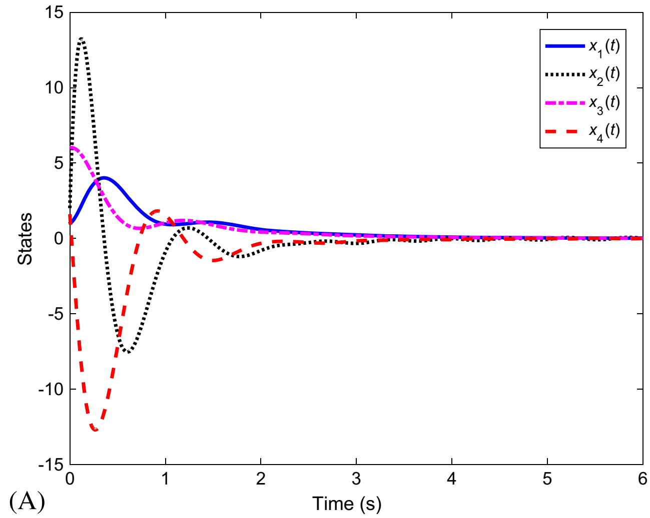


FIGURE 3 Saturated control signal response for uncertain nonlinear one-link flexible robotic system



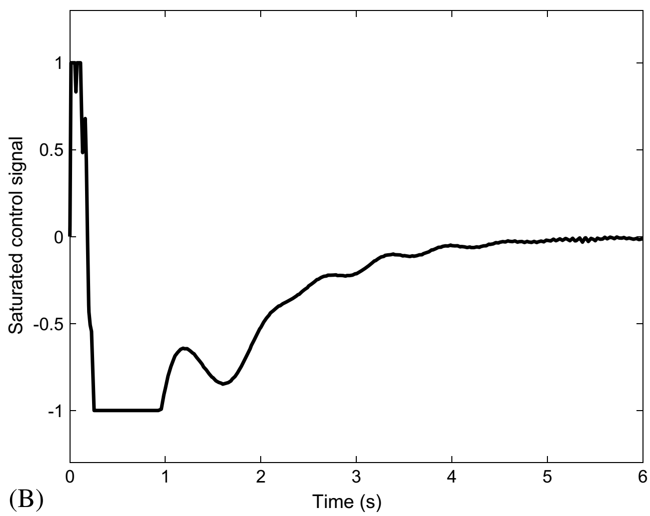


FIGURE 4 Closed-loop response of uncertain nonlinear system under double initial condition. A, Closed-loop state trajectories; B, Saturated control signal [Colour figure can be viewed at wileyonlinelibrary.com]

converges to a region in the neighborhood of the origin in the presence of saturation nonlinearity, parametric norm-bounded uncertainties, and input and output disturbances, which validates the usefulness of the proposed windup compensation-based control technique. Figure 3 shows the corresponding saturated control signal plot. The control signal recovers within 1 second from the saturation nonlinearity and then converges in the presence of disturbances. It shows the ability of the proposed control approach to recover from the windup phenomenon caused by the saturation nonlinearity.

To check the performance of the proposed control strategy against saturation nonlinearity, we test our control system under large initial conditions and more complex disturbance signal. The response of the closed-loop uncertain nonlinear one-link flexible robotic system with a large initial condition (two times larger than the previous case) $\left[x_m(0)\ x_{wm}(0)\ x_L(0)\ x_{wL}(0)\right]^T = \left[1\ 2\ 6\ 1.6\right]^T$ under same external disturbances is demonstrated in Figure 4A. The corresponding saturated control signal is depicted in Figure 4B. It has been observed that the stabilization of the closed-loop response is guaranteed, without any input saturation effect. As can be observed from Figure 4A, the closed-loop system responses of states converge in the presence of saturation nonlinearity, parametric norm-bounded uncertainties, and external disturbance. The saturated control signal recovers from saturation around 1 second and regulates states of the uncertain nonlinear plant. It further validates the effectiveness of the proposed windup compensation-based control technique.

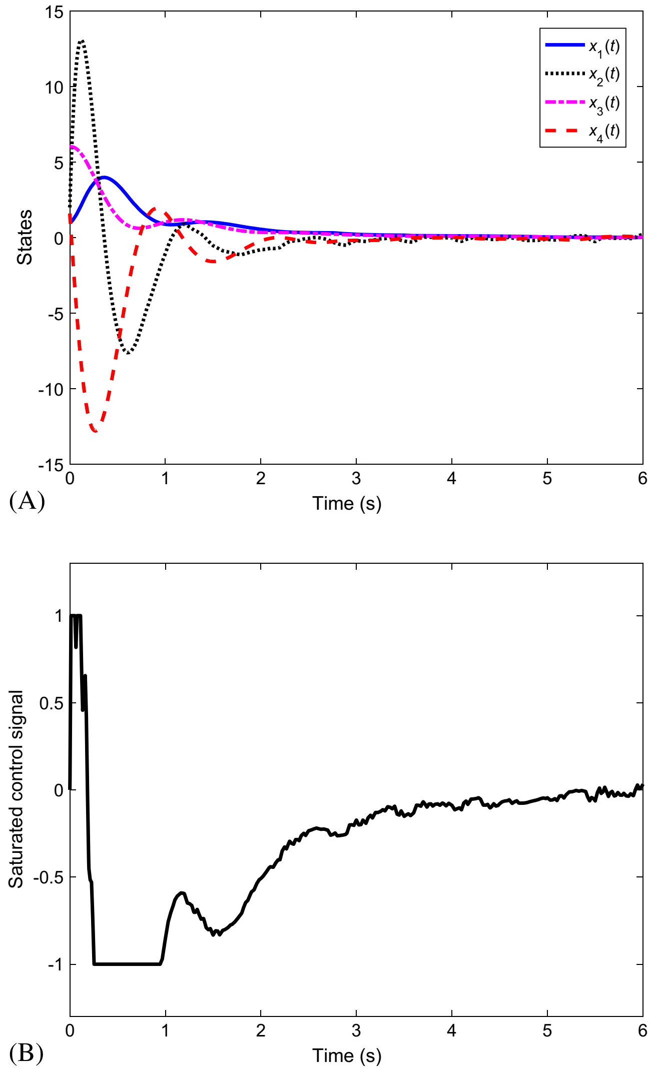


FIGURE 5 Closed-loop response of uncertain nonlinear system under double initial condition and random disturbance of large amplitude. A, Closed-loop state trajectories; B, Saturated control signal [Colour figure can be viewed at wileyonlinelibrary.com]

Now, we investigate the closed-loop response and saturated control signal by considering large initial conditions and more practical large random disturbance simultaneously. We select the initial condition as $\left[x_m(0)\ x_{wm}(0)\ x_L(0)\ x_{wL}(0) \right]^T = \left[1\ 2\ 6\ 1.6 \right]^T$ and apply an external disturbance of a uniformly distributed random signal belonging to [−3.5 3.5]. The resultant plots for closed-loop states and saturated control signal are provided in Figure 5. It is observed from Figure 5A that the closed-loop system response converges in the neighborhood of the origin in the presence of saturation nonlinearity with a saturation limit of $\bar{v} = \pm 1$, parametric norm-bounded uncertainties, and large random external disturbance of a uniformly distributed random signal of magnitude ±3.5. The saturated control signal is shown in Figure 5B, which recovers from the saturation and adjusts in accordance with the random disturbance for controlling the uncertain nonlinear system. It again reveals that the proposed windup compensation-based control technique is impregnable against saturation, uncertainties, and external disturbance because all the states converge and control signal recovers from saturation nonlinearity without generating the windup effects.

5 | CONCLUSIONS

A novel technique for the simultaneous design of a robust nonlinear controller and anti-windup compensation for uncertain nonlinear systems with actuator saturation, parametric uncertainties, and exogenous $\mathcal{L}_2$ bounded disturbances has been investigated in this study. The system considered in this paper is presumed to have locally Lipschitz nonlinearities, time-varying uncertainties, and external norm-bounded disturbances. Several design conditions have been derived to attain the AWC-based robust nonlinear controller for uncertain nonlinear systems by employing the Lyapunov functional, reformulated Lipschitz property, uncertainty bounds, LPV theory, modified sector condition, convex optimization procedure, and $\mathcal{L}_2$ gain minimization. The proposed conditions can be resolved by employing the ILMI optimization procedure. The proposed multiobjective AWC-based dynamic robust nonlinear controller guarantees the mitigation of saturation effects, robustness against time-varying parametric uncertainties, the asymptotic stability of the closed-loop system under zero external disturbances, and the attenuation of disturbance effects under nonzero external disturbances. Applications to a nonlinear system and an uncertain nonlinear one-link flexible robotic system verified the effectiveness of the proposed AWC-based dynamic robust nonlinear control scheme.

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ORCID

Muhammad Rehan https://orcid.org/0000-0002-9908-3971
Choon Ki Ahn https://orcid.org/0000-0003-0993-9658
Keum-Shik Hong https://orcid.org/0000-0002-8528-4457

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